Bio-thermal convection caused by combined effects of swimming of oxytactic bacteria and inclined temperature gradient in a shallow fluid layer

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Abstract

Purpose – The aim of this paper is to investigate the onset of bio-thermal convection in a shallow fluid layer; the convection is thus driven by the combined effect of swimming of oxytactic microorganisms and inclined temperature gradient.

Design/methodology/approach – Linear stability analysis of the basic state is performed; the numerical problem is solved using the collocation method.

Findings – The most interesting outcome of this analysis is the correlation between three Rayleigh numbers, two traditional, "thermal" Rayleigh numbers, which are associated with the vertical and horizontal temperature gradients in the fluid layer, and the bioconvection Rayleigh number, which is associated with the density variation induced by the upswimming of microorganisms.

Research limitations/implications – Further research should address the application of weakly nonlinear analysis to this problem.

Practical implications – The increase of the horizontal thermal Rayleigh number stabilizes the basic flow. The effect of increasing the horizontal thermal Rayleigh number is to distort the basic temperature profile away from the linear one. The increase of the Schmidt number stabilizes the basic flow. The increase of the Prandtl number first causes the bioconvection Rayleigh number to decrease and then to increase.

Originality/value – To the best of the authors' knowledge, this is the first research dealing with the effect of inclined temperature gradient on the stability of bioconvection.

Keywords Convection, Temperature distribution, Thermal stability, Fluids, Microbiology

Paper type Research paper

Nomenclature

b	chemotaxis constant	$\tilde{C}_{\min}$	minimum dimensional oxygen concentration that microorganisms need in order to be active
C	dimensionless oxygen concentration		
$\tilde{C}$	dimensional oxygen concentration defined in Equation (8)	D_N	diffusivity of microorganisms
$\tilde{C}_0$	free-surface dimensional oxygen concentration	D_C	diffusivity of oxygen
		g	gravity

G	rescaled dimensionless disturbance amplitude of oxygen concentration, Ξ/ς	$\tilde{T}_0$	ambient temperature
H	depth of the horizontal fluid layer	$\tilde{u}, \tilde{\mathrm{v}}, \tilde{w}$	$\tilde{x}, \tilde{y}, \tilde{z}$-velocity components, respectively
$\hat{\mathrm{H}}$	heaviside step function	U_s, V_s, W_s	x, y, z-components of the dimensionless steady-state velocity, $\tilde{U}_s H\rho/\mu$, $\tilde{V}_s H\rho/\mu$, $\tilde{W}_s H\rho/\mu$, respectively
$\tilde{\mathbf{j}}$	total flux of microorganisms due to macroscopic convection of the fluid, self-propelled swimming of microorganisms, and diffusion of micro organisms, $\tilde{n}\tilde{\mathbf{v}} + \tilde{n}\tilde{\mathbf{W}} - D_N\nabla\tilde{n}$	$\tilde{U}_s, \tilde{V}_s, \tilde{W}_s$	$\tilde{x}, \tilde{y}, \tilde{z}$-components of the steady-state velocity, respectively
k, l	dimensionless wavenumbers in the x- and y-directions, respectively	$\tilde{\mathbf{v}}$	fluid convection velocity vector $(\tilde{u}, \tilde{\mathrm{v}}, \tilde{w})$
$\hat{\mathbf{k}}$	vertically downward unit vector	W_c	parameter characterizing the maximum cell swimming speed
m	dimensionless wavenumber, $m^2 = k^2 + l^2$	$\tilde{\mathbf{W}}$	average microorganisms' swimming velocity relative to the fluid, defined in Equation (7)
n	dimensionless number density of motile microorganisms, $\tilde{n}/n_0$	$\tilde{x}, \tilde{y}, \tilde{z}$	Cartesian coordinates ($\tilde{z}$ is the downward vertical coordinate)
$\tilde{n}$	number density of motile microorganisms	x, y, z	dimensionless Cartesian coordinates, $\tilde{x}/H, \tilde{y}/H$, and $\tilde{z}/H$, respectively
$\tilde{n}_0$	average number density of the micro-organisms, defined in Equation (20)		
p	excess pressure (above hydrostatic)	*Greek symbols*	
Pe	bioconvection Péclet number, bW_c/D_N	α	thermal diffusivity of the suspension
Pr	Prandtl number, $\mu/\alpha\rho_w$	β	volumetric expansion coefficient of the fluid
Ra_B	bioconvection Rayleigh number, $gH^3\Delta\rho\theta\tilde{n}_0/\mu D_N$	β_H	horizontal temperature gradient
Ra_H	horizontal thermal Rayleigh number, $g\beta H^4\beta_H\rho_w/\mu\alpha$	γ	rate of consumption of oxygen by the microorganisms
Ra_V	vertical thermal Rayleigh number, $g\beta H^3\Delta T\rho_w/\mu\alpha$	δ	diffusion parameter, D_C/D_N
Sc	Schmidt number, $\mu/D_N\rho_w$	ΔC	difference between the free-surface and minimum oxygen concentrations, $\tilde{C}_0 - \tilde{C}_{\min}$
t	dimensionless time, $\tilde{t}\mu/\rho H^2$		
$\tilde{t}$	time	ΔT	temperature difference between the lower and top surfaces measured at the same $(\tilde{x}, \tilde{y})$ position
T	dimensionless temperature, $(\tilde{T} - \tilde{T}_0)/\Delta T$		
$\tilde{T}$	temperature		

$\Delta\rho$	density difference, $\rho_{cell} - \rho_w$	σ	dimensionless disturbance frequency
θ	average volume of a microorganism	ς	dimensionless rate of consumption of oxygen by the microorganisms, $\gamma n_0 H^2 / D_C \Delta C$
Θ	rescaled dimensionless disturbance amplitude of temperature, $\Omega(\alpha/D_N)$	ϖ	dimensionless parameter, Peς
μ	dynamic viscosity		
ρ_w	density of water	*Subscript*	
ρ_{cell}	density of cells	s	steady state

1. Introduction

Bioconvection is the spontaneous formation of macroscopic fluid patterns, such as falling plumes. It results from complex interaction of phenomena at different physical scales. The process is driven by the directional locomotion of self-propelled microorganisms that are denser than water; the swimming of each individual microorganism is a mesoscale physical phenomenon. The macroscopic density gradient caused by swimming of a large number of microorganisms (whose swimming velocity typically has an upward component) induces convection instability that results in the formation of a spatially periodic macroscopic fluid circulation.

The purpose of this paper is to investigate the interaction between bioconvection caused by motile oxytactic microorganisms, such as bacteria *Bacillus subtilis*, and natural convection caused by an inclined temperature gradient. The goal is to obtain a steady-state solution for a system subjected to an inclined temperature gradient and then investigate how the inclined temperature gradient affects the stability of this steady-state solution. The theory of oxytactic bioconvection was developed in Hillesdon *et al.* (1995), Hillesdon and Pedley (1996), and Metcalfe and Pedley (1998, 2001). Becker *et al.* (2004) extended this theory to oxytactic bioconvection in porous media and Avramenko and Kuznetsov (2005) investigated the onset of oxytactic bioconvection in superimposed fluid and porous layers. Kuznetsov (2005a, b) introduced the theory of bio-thermal convection caused by the combined effect of vertical temperature gradient and the upswimming of oxytactic microorganisms in clear fluids, Kuznetsov (2006a) extended this theory to bio-thermal convection in porous media, and Kuznetsov (2006b) investigated the effect of high-frequency vibration on the onset of oxytactic bioconvection.

The instability problem of natural convection in a fluid layer of finite depth induced by the inclined temperature gradient was first investigated by Weber (1973, 1978). Nield (2004) reformulated the linear stability analysis for the inclined temperature gradient problem to lift the restrictions on the range of the Prandtl number. Recent advances in convection problems associated with an inclined temperature gradient are documented in Kaloni and Lou (2002, 2005), who considered extensions of this problem to Oldroyd-B and viscoelastic fluids; and Shklyaev and Nepomnyashchy (2004), who studied the stability of thermocapillary flows generated by an inclined temperature gradient. Recently, Avramenko and Kuznetsov (2010) investigated a combined bioconvection and thermal instability problem in a horizontal layer of finite depth with a basic temperature gradient inclined to the vertical, so that the basic flow (driven by the horizontal component of temperature gradient) is a single cell – the Hadley circulation. In Avramenko and Kuznetsov (2010), the suspension consisting of gyrotactic motile microorganisms was considered. The purpose of this paper is to investigate the onset of

bio-thermal convection in a suspension of oxytactic microorganisms, such as bacteria *Bacillus subtilis*, in a fluid layer with an inclined temperature gradient.

It is assumed that the inclined temperature gradient is moderate, so that the temperature variation does not kill the microorganisms and does not affect their oxytactic behavior. The dependence of the solubility of oxygen in water on the temperature is neglected. Oxytactic bacteria require oxygen for their metabolism and swim up the oxygen gradients. They require a minimum oxygen concentration, $\tilde{C}_{min}$, in order to be active. Since this paper deals with a shallow layer, it is assumed that even at the bottom of the layer the oxygen concentration is larger than the minimum concentration.
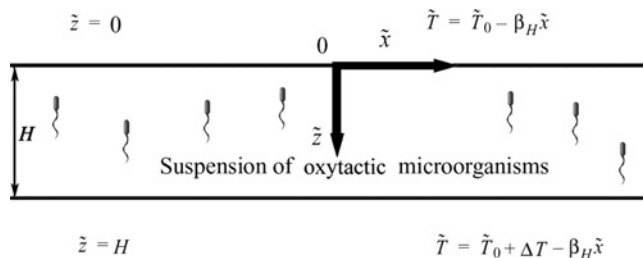
2. Governing equations
The model for bioconvection used in this study is based on that presented in Hillesdon *et al.* (1995), Hillesdon and Pedley (1996), and Metcalfe and Pedley (1998, 2001). This model is supplemented by an energy equation and a buoyancy term in the momentum equation that results from the inclined temperature gradient. The geometry of the problem is similar to that considered in Nield (2004) (see Figure 1). The suspension of oxytactic microorganisms is confined in a shallow horizontal box with small height-to length and height-to-width aspect ratios. The two horizontal walls of the box are at a distance H apart. A Cartesian coordinate system $(\tilde{x}, \tilde{y}, \tilde{z})$ (with the $\tilde{z}$-axis vertically downward) is chosen such that its origin is in the middle of the box. A linear horizontal temperature gradient is imposed in the $\tilde{x}$-direction, and a constant temperature difference is imposed between the two horizontal walls. The situation described above can be realized in a carefully planned lab experiment. The Boussinesq approximation is utilized. Under these assumptions, the governing equations are presented as:

$$\rho_w \left(\frac{\partial \tilde{\mathbf{v}}}{\partial \tilde{t}} + (\tilde{\mathbf{v}}\nabla)\tilde{\mathbf{v}} \right) = -\nabla p + \mu\nabla^2\tilde{\mathbf{v}} + \hat{\mathbf{k}}(n\theta\Delta\rho g - \rho_w g\beta(\tilde{T} - \tilde{T}_0)) \qquad (1)$$

$$\text{div}(\tilde{\mathbf{v}}) = 0 \qquad (2)$$

$$\frac{\partial \tilde{n}}{\partial \tilde{t}} = -\text{div}(\tilde{\mathbf{j}}) \qquad (3)$$

$$\frac{\partial \tilde{T}}{\partial \tilde{t}} + (\tilde{\mathbf{v}}\nabla)\tilde{T} = \alpha\nabla^2\tilde{T} \qquad (4)$$

Figure 1.
Definition sketch

$$\frac{\partial C}{\partial \tilde{t}} + (\tilde{\mathbf{v}}\nabla)C = D_C\nabla^2 C - \frac{\gamma\tilde{n}}{\Delta C} \tag{5}$$

where the total flux of microorganisms due to macroscopic convection of the fluid, self-propelled swimming of microorganisms, and diffusion of microorganisms is given by:

$$\tilde{\mathbf{j}} = \tilde{n}\tilde{\mathbf{v}} + \tilde{n}\tilde{\mathbf{W}} - D_N\nabla\tilde{n} \tag{6}$$

where D_N is the diffusivity of microorganisms; α is the thermal diffusivity of the suspension; g is the gravity; $\tilde{n}$ is the number density of motile microorganisms; $\tilde{T}$ is the temperature; $\tilde{T}_0$ is the ambient temperature; p is the excess pressure (above hydrostatic); $\hat{\mathbf{k}}$ is the vertically downward unit vector; $\tilde{t}$ is the time; $\tilde{\mathbf{v}}$ is the fluid convection velocity vector with components $\tilde{u}$, $\tilde{v}$, $\tilde{w}$; θ is the average volume of a microorganism; β is the volumetric expansion coefficient of the fluid; μ is the dynamic viscosity, assumed to be approximately the same as that of water; $\Delta\rho = \rho_{cell} - \rho_w$ is the density difference; ρ_w is the density of water; ρ_{cell} is the density of cells; D_C is the diffusivity of oxygen; and $-\gamma\tilde{n}/\Delta C$ describes the consumption of oxygen by the microorganisms in the fluid.

The average directional swimming velocity of a microorganism is approximated as (Hillesdon and Pedley, 1996):

$$\tilde{\mathbf{W}} = bW_c\hat{\mathrm{H}}(C)\nabla C \tag{7}$$

where b is the chemotaxis constant (m) and W_c is a parameter characterizing the maximum cell swimming speed (m/s) (the product bW_c is assumed to be constant). The dimensionless oxygen concentration, C, in Equations (5) and (7) is defined as:

$$C = \frac{\tilde{C} - \tilde{C}_{min}}{\tilde{C}_0 - \tilde{C}_{min}} = \frac{\tilde{C} - \tilde{C}_{min}}{\Delta C} \tag{8}$$

where $\tilde{C}$ is the dimensional oxygen concentration, $\tilde{C}_0$ is the free-surface dimensional oxygen concentration, and $\tilde{C}_{min}$ is the minimum dimensional oxygen concentration that microorganisms need in order to be active. Since for the shallow layer $C > 0$ throughout the layer thickness, the Heaviside step function, $\hat{\mathrm{H}}(C)$, in Equation (7) is equal to unity.

3. Boundary conditions
As in Nield (2004), the ratio of the height to the length of the layer is assumed to be sufficiently small so that fluid motion in the horizontally central part of the box is not affected by lateral end effects.

If the top surface is rigid, the following conditions are satisfied there:

$$\tilde{\mathbf{v}} = 0, \quad \tilde{T} = \tilde{T}_0 - \beta_H\tilde{x}, \quad C = 1, \quad \tilde{\mathbf{j}}\cdot\hat{\mathbf{k}} = 0 \text{ at } \tilde{z} = 0 \tag{9}$$

where β_H is the constant horizontal temperature gradient.

If the top surface is stress-free, the first equation in (9) must be replaced with the following two equations:

$$\frac{\partial^2 (\mathbf{vk})}{\partial \tilde{z}^2} = 0, \quad \tilde{\mathbf{v}} \cdot \hat{\mathbf{k}} = 0 \tag{10}$$

The last equation in (9) ($\tilde{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0$) can be recast as:

$$\tilde{n} b W_c \frac{\partial C}{\partial \tilde{z}} - D_N \frac{\partial \tilde{n}}{\partial \tilde{z}} = 0 \tag{11}$$

At the bottom of the layer (assumed to be rigid) the following conditions are satisfied:

$$\tilde{\mathbf{v}} = 0, \quad \tilde{T} = \tilde{T}_0 + \Delta T - \beta_H \tilde{x}, \quad \frac{\partial C}{\partial \tilde{z}} = 0, \quad \tilde{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0 \quad \text{at } \tilde{z} = 1 \tag{12}$$

The last equation in (12) ($\tilde{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0$), accounting for $\partial C/\partial \tilde{z} = 0$, can be recast as:

$$\frac{\partial \tilde{n}}{\partial \tilde{z}} = 0 \tag{13}$$

4. Steady-state solution

From Equations (1)-(5) and the fact that the layer is infinite in the horizontal directions it follows that at steady state the number density of microorganisms, $\tilde{n}_s$, is a function of the vertical coordinate only; under this condition the steady-state solutions for the temperature and velocity components are, respectively (Nield, 2004):

$$T_s = z - \frac{1}{2} - \frac{\text{Ra}_H^2}{\text{Ra}_V} x - \frac{\text{Ra}_H^2}{5{,}760 \text{Ra}_V} \left(7 \left(z - \frac{1}{2} \right) - 40 \left(z - \frac{1}{2} \right)^3 + 48 \left(z - \frac{1}{2} \right)^5 \right), \tag{14}$$

$$U_s = -\frac{\text{Ra}_H}{24 \text{Pr}} \left(\left(z - \frac{1}{2} \right) - 4 \left(z - \frac{1}{2} \right)^3 \right), \quad V_s = W_s = 0 \tag{15}$$

where the dimensionless $\tilde{x}$ and $\tilde{z}$ coordinates, steady-state temperature, and velocity components are defined by the following equations, respectively:

$$x = \tilde{x}/H, \quad z = \tilde{z}/H, \quad T_s = \frac{\tilde{T}_s - \tilde{T}_0}{\Delta T}, \quad U_s = \frac{\tilde{U}_s H \rho}{\mu}, \quad V_s = \frac{\tilde{V}_s H \rho}{\mu}, \quad W_s = \frac{\tilde{W}_s H \rho}{\mu} \tag{16}$$

and the vertical thermal Rayleigh number, horizontal thermal Rayleigh number, and Prandtl number are defined, respectively, as:

$$\text{Ra}_V = \frac{g \beta H^3 \Delta T \rho_w}{\mu \alpha}, \quad \text{Ra}_H = \frac{g \beta H^4 \beta_H \rho_w}{\mu \alpha}, \quad \text{Pr} = \frac{\mu}{\alpha \rho_w} \tag{17}$$

The steady-state solutions for C_s and $\tilde{n}_s$ are found in Hillesdon *et al.* (1995) as follows:

$$C_s = 1 - \frac{2}{\text{Pe}} \ln \frac{\cos(A_1(1-z)/2)}{\cos(A_1/2)} \tag{18}$$

$$n_s = \frac{A_1}{2\text{Pe}\varsigma} \sec^2(A_1(1-z)/2) \tag{19}$$

where,

$$\text{Pe} = \frac{bW_c}{D_N}, \quad n_s = \frac{\tilde{n}_s}{\tilde{\boldsymbol{n}}_0}, \quad \varsigma = \frac{\gamma\tilde{\boldsymbol{n}}_0 H^2}{D_C \Delta C}, \quad \tilde{\boldsymbol{n}}_0 = \frac{1}{H}\int_0^H \tilde{n}\, d\tilde{z}, \tag{20}$$

where $\tilde{\boldsymbol{n}}_0$ is the average number density of the microorganisms. The constant A_1 is the smallest positive root of the following transcendental equation:

$$\tan\left(\frac{1}{2}A_1\right) = \frac{\text{Pe}\varsigma}{A_1} \tag{21}$$

5. Linear stability analysis
The perturbations are introduced as follows:

$$[n, T, \mathbf{v}, p, C] = [n_s(z), T_s(x, z), \mathbf{v}_s(U_s(z), 0, 0), p_s(z), C_s(z)]$$
$$+ [N(z), \Omega(z), \mathbf{V}(U(z), V(z), W(z)), P(z), \Xi(z)] \exp[i(kx + ly - \sigma t)] \tag{22}$$

where,

$$y = \frac{\tilde{y}}{H}, \quad t = \frac{\tilde{t}\mu}{\rho H^2} \tag{23}$$

k and l are the dimensionless wavenumbers in the x- and y-directions, respectively, and σ is the dimensionless disturbance frequency.

Substituting Equation (22) into the dimensionless version of Equations (1)-(5) and linearizing results in equations for the amplitudes U, V, W, P, N, Ω, and Ξ. The elimination of P, U, and V from these equations for amplitudes gives the following equations for the remaining amplitudes W, G (G is related to Ξ by first equation in (28)), N, and Θ (Θ is related to Ω by third equation in (28)):

$$\frac{d^4 W}{dz^4} + [-2m^2 + i(\sigma - kU_s)]\frac{d^2 W}{dz^2} + \left[m^4 + i\left(m^2 kU_s + k\frac{d^2 U_s}{dz^2} - m^2\sigma\right)\right]W$$
$$= m^2(\text{Ra}_B N - \text{Ra}_V \Theta) \tag{24}$$

$$\left[\frac{d^2}{dz^2} - \varpi\frac{dG_s}{dz}\frac{d}{dz} - \left(m^2 + i\text{Sc}(kU_s - \sigma) + \varpi\frac{d^2 G_s}{dz^2}\right)\right]N$$
$$= n_s\varpi\left(\frac{d^2 G}{dz^2} - m^2 G\right) + \frac{dn_s}{dz}\left(W + \varpi\frac{dG}{dz}\right) \tag{25}$$

$$\left[\frac{d^2}{dz^2} - (m^2 + i\Pr(kU_s - \sigma))\right]\Theta = \frac{dT_s}{dz}W \tag{26}$$

$$\left[\delta\frac{d^2}{dz^2} - (\delta m^2 + i\mathrm{Sc}(kU_s - \sigma))\right]G = \delta N + W\frac{dG_s}{dz} \tag{27}$$

where,

$$G = \frac{\Xi}{\varsigma}, \quad G_s = \frac{C_s}{\varsigma}, \quad \Theta = \Omega\frac{\alpha}{D_N}, \quad \mathrm{Ra}_B = \frac{gH^3\Delta\rho\theta\tilde{n}_0}{\mu D_N} \tag{28}$$

$$\delta = \frac{D_C}{D_N}, \quad \varpi = \mathrm{Pe}\varsigma, \quad m^2 = k^2 + l^2, \quad \mathrm{Sc} = \frac{\mu}{D_N\rho_w} \tag{29}$$

where Ra_B is the bioconvection Rayleigh number and Sc is the Schmidt number.

Steady-state solutions in Equations (24)-(28) are those given by Equations (14), (15), (18), and (19).

If both the lower and top boundaries are assumed rigid, Equations (24)-(27) must be solved subject to the following boundary conditions:

$$W = 0, \quad \frac{dW}{dz} = 0, \quad \Theta = 0, \quad G = 0, \quad \mathrm{Pe}\left(n_s\frac{dG}{dz} + N\frac{dG_s}{dz}\right) = \frac{dN}{dz} \quad \text{at } z = 0 \tag{30}$$

$$W = 0, \quad \frac{dW}{dz} = 0, \quad \Theta = 0, \quad \frac{dG}{dz} = 0, \quad \frac{dN}{dz} = 0 \quad \text{at } z = 1 \tag{31}$$

If the top boundary is stress-free, second equation in (30) ($dW/dz = 0$) must be replaced with the following equation:

$$\frac{d^2W}{dz^2} = 0 \tag{32}$$

6. Numerical method and validation
The collocation method (Fletcher, 1984) is used for obtaining the numerical solution of the eigenvalue problem for Equations (24)-(27). For the case of the no-slip condition at the top surface (boundary conditions for this case are given by Equations (30) and (31)) the basis functions are:

$$W = \sum_{j=1}^{M} a_j z^{j+1}(1-z)^{j+1},$$

$$\Theta = \sum_{j=1}^{M} b_j z^j(1-z)^j,$$

$$N = 1 + \chi \sum_{j=1}^{M} c_j z^j (1 - z/2)^j,$$

$$G = \sum_{j=1}^{M} d_j z^j (1 - z/2)^j,$$ (33a, b, c, d)

where,

$$\chi = \frac{A_1 (A_1 - \sin A_1)}{1 + \cos A_1}$$ (34)

($M = 100$ is used in the numerical implementation of the code.)

For the case of the stress-free top boundary (see Equation (32)), Equation (33a) must be replaced with:

$$W = \sum_{j=1}^{M} a_j (z - 3z^3 + 2z^4)^{j+1}$$ (35)

The solution of the eigenvalue problem leads to the following dependence:

$$Ra_B = Ra_B(m, k, Ra_V, Ra_H, \varpi, Sc, Pr, \delta, \sigma)$$ (36)

δ	ϖ	Hillesdon and Pedley (1996)		Present paper	
		m_{cr}	$Ra_{B,cr}$	m_{cr}	$Ra_{B,cr}$
1	0.05	1.37	10,200	1.38	10,237
1	1	1.58	625	1.58	625
1	10	1.90	200	1.90	200

Notes: Critical wavenumber obtained in this paper with that reported in Table III of Hillesdon and Pedley (1996), computations are performed for an isothermal layer with a stress-free top boundary

Table I.
Comparison of the critical bioconvection Rayleigh number and wavenumber in this paper

Parameter	Definition	Range or typical value
Sc	$(\mu/\rho_w)/D_N$	7,700
Pr	$(\mu/\rho_w)/\alpha$	1-11
ς	$\gamma \tilde{n}_0 H^2 / D_C \Delta C$	43.75
Pe	$W_c H / D_N$	0-37.5
ϖ	Pe ς	0-1,640
δ	D_C / D_N	16
Ra_V	$g \beta H^3 \Delta T \rho_w / \mu \alpha$	0-284,375
Ra_H	$g \beta H^4 \beta_H \rho_w / \mu \alpha$	0-7,200
Ra_B	$g H^3 \Delta \rho \theta \tilde{n}_0 / \mu D_N$	15,625

Table II.
Values of dimensionless parameters
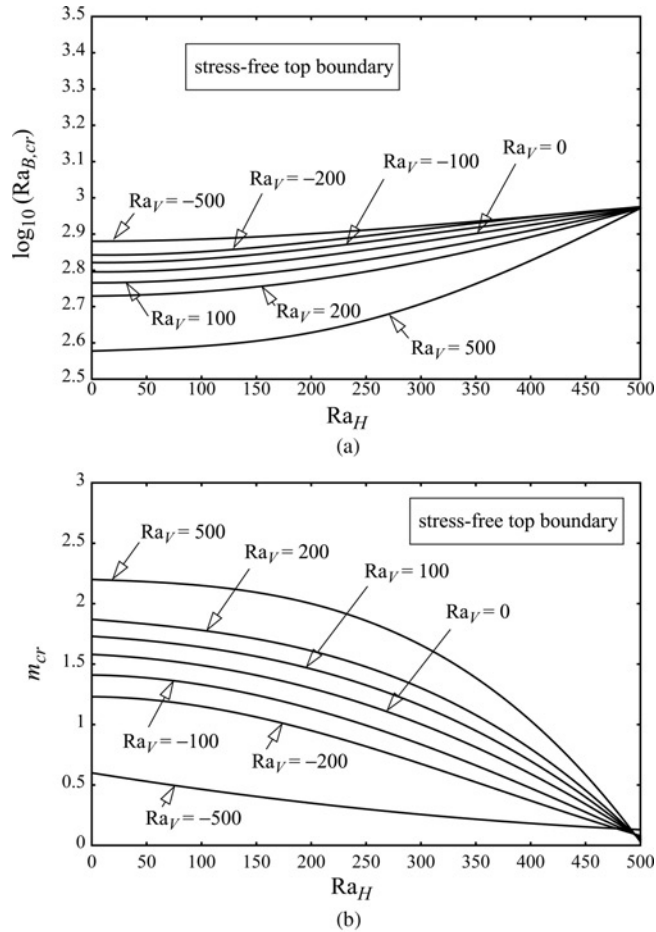
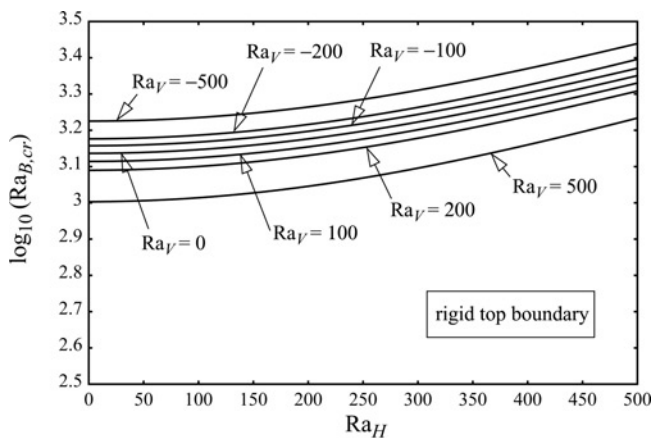**Figure 2.**
The stress-free upper
boundary case

Notes: Effect of the horizontal Rayleigh numbers on the critical
bioconvection Rayleigh number, Ra$_{B,cr}$ (a) and on the critical
wavenumber, m_{cr} (b)

for the bioconvection Rayleigh number. The critical bioconvection Rayleigh number is
then obtained as:

$$\text{Ra}_{B,cr} = \min\{\text{Ra}_B(m, k, \text{Ra}_V, \text{Ra}_H, \varpi, \text{Sc}, \text{Pr}, \delta, \sigma_i = 0)\} \qquad (37)$$

where σ_i is the imaginary part of σ.

The collocation method for this eigenvalue problem is implemented utilizing the
Matlab package. For validating the numerical code data obtained in Hillesdon and
Pedley (1996) are used. Table I presents a comparison between the values of Ra$_{B,cr}$ and
m_{cr} that follow from Equations (24)-(27) and data of Hillesdon and Pedley (1996). Table I
is for the case of the stress-free top surface (the only boundary condition considered in
the above reference). There is an excellent agreement between the present
computational results and data obtained in Hillesdon and Pedley (1996).
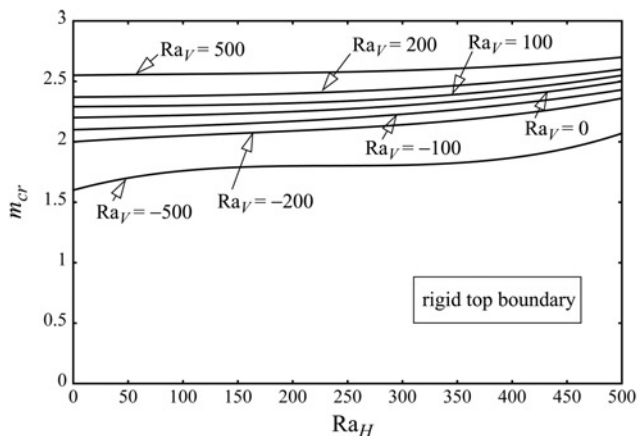
Figure 3.
Same as Figure 2, the
rigid upper boundary case

7. Results and discussion
Typical values of the dimensionless parameters for a soil bacterium *Bacillus subtilis*
are estimated in Hillesdon and Pedley (1996) as follows: $Pe = 15H$, $\varsigma = 7H^2$, and
$\delta = 16$, where the layer depth, H, must be given in mm. A typical depth of a shallow
layer in experiments described in Hillesdon *et al.* (1995) was 2.5 mm. For estimating the
vertical and horizontal temperature gradients a 10 °C temperature difference in both
vertical and horizontal direction is assumed; the characteristic length in the horizontal
direction is assumed to be 100 mm. The values of Ra_V and Ra_H should be understood as
estimates of the maximum safe values of these parameters so that the temperature
variation does not kill microorganisms; smaller values of Ra_V and Ra_H can be easily
realized in a lab experiment. Using parameter values given in Hillesdon and Pedley
(1996) and assuming that $\tilde{n}_0 = 10^{15}$ cell/m^3, an estimate of the range or a typical
values of dimensionless parameters is given in Table II. Since the aim of this paper is
not to study a particular biological system (not enough experimental data are available
at this point to accomplish this task) but rather to investigate the trends, $Ra_V =
Ra_H = 100$, $\varpi = 1$, and $Pr = Sc = 1$, and $\delta = 1$ are adopted as a basic case, and then
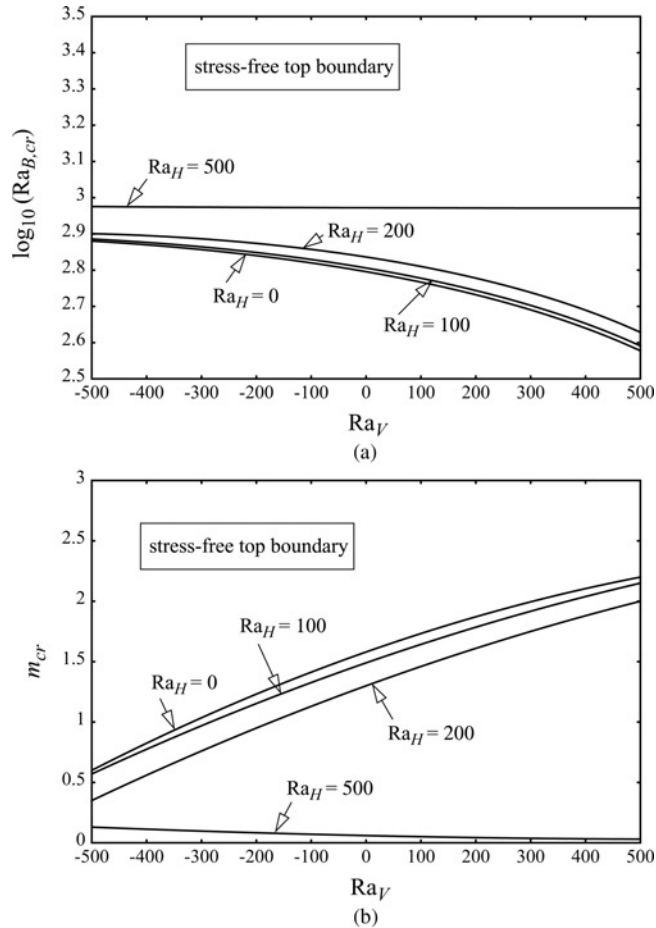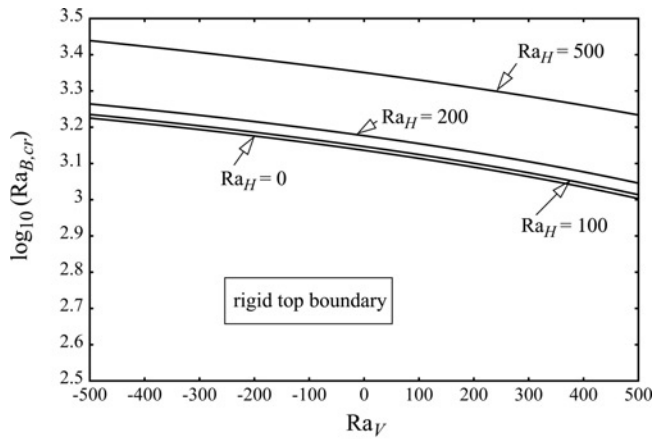
Figure 4.
The stress-free upper
boundary case

Notes: Effect of the vertical Rayleigh numbers on the critical
bioconvection Rayleigh number, $Ra_{B,cr}$ (a) and on the critical
wavenumber, m_{cr} (b)

the values of Ra_V, Ra_H, ϖ, Pr, Sc, and δ are varied around the basic case values in
Figures 2-8.

Computational results for all parameter values investigated in this research indicate
that the marginal state is stationary. This is in agreement with Hillesdon and Pedley
(1996) who found no oscillatory marginal states for shallow chambers. Computational
results presented in Figures 2 and 4 (stress-free top boundary) and Figures 3 and 5
(rigid top boundary) demonstrate the effects of the horizontal and vertical Rayleigh
numbers on the bioconvection Rayleigh number. These results are computed for
$\varpi = 1$, $\delta = 1$, and Pr = Sc = 1. Figures 2a and 3a indicate that for both types of
boundary conditions at the top surface the increase of the horizontal Rayleigh number
stabilizes the system (increases the critical value of the bioconvection Rayleigh
number). This suggests that the horizontal temperature gradient has the stabilizing
effect on the system. This is explained as follows. The effect of increasing the

(a)
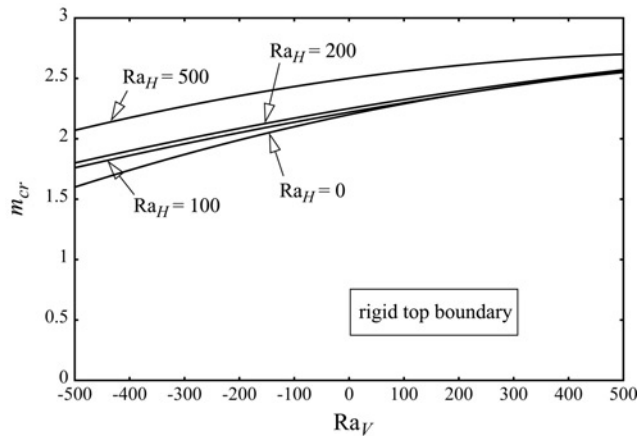
(b)

Figure 5.
Same as Figure 4, the
rigid upper boundary case

horizontal thermal Rayleigh number is to distort the basic temperature profile away
from the linear one; this produces the stabilizing effect because the destabilizing
vertical temperature gradient is decreased in magnitude in the bulk of the fluid.
Figures 2b and 3b indicate that the increase of the horizontal Rayleigh number
decreases the critical wavenumber for the stress-free top boundary and slightly
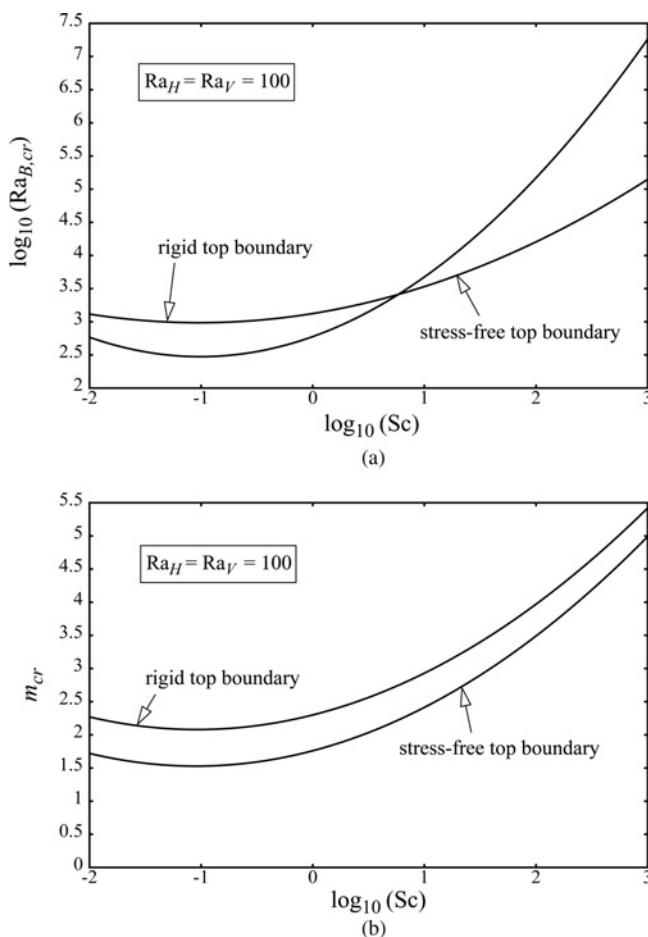increases the critical wavenumber for the rigid top boundary.

Figures 4a and 5a indicate that for both types of boundary conditions at the
top surface and for all values of Ra_H considered in these computations the critical
bioconvection Rayleigh number, $Ra_{B,cr}$, decreases with the increase of Ra_V. A similar
trend is observed in Kuznetsov (2005a, b), where the case with no horizontal
temperature gradient is investigated. This is as expected because increasing the
vertical Rayleigh number means the increase of destabilizing effect of the vertical
temperature gradient (positive values of the vertical Rayleigh number correspond to
the negative vertical temperature gradient whose effect is destabilizing). The obtained
results indicate that the boundary condition at the top surface strongly affects the
behavior of the critical bioconvection Rayleigh number. For the stress-free top
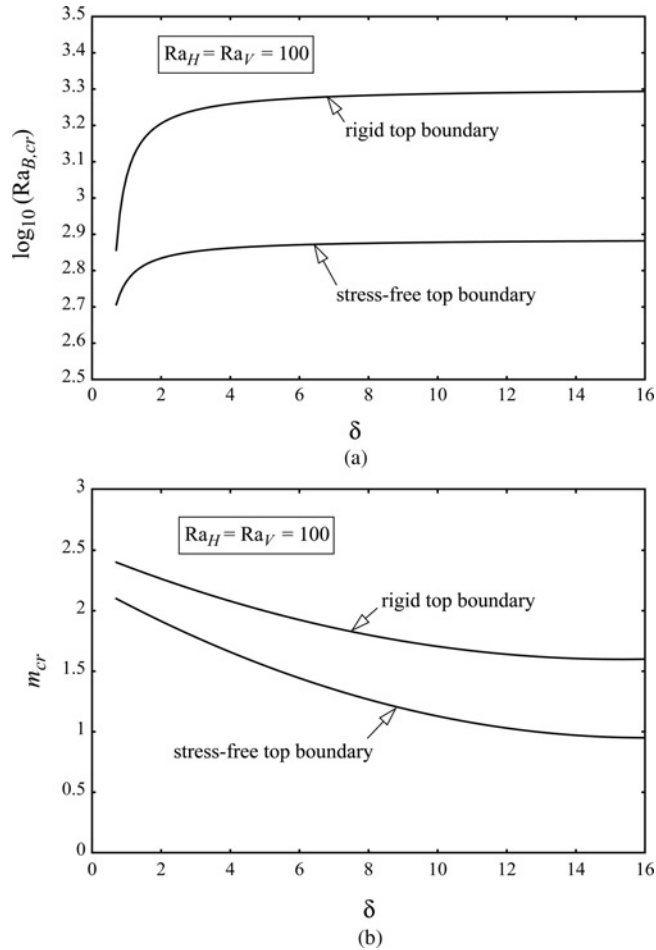
Figure 6.
Effect of δ on the critical bioconvection Rayleigh number, $\mathrm{Ra}_{B,cr}$ (a) and on the critical wavenumber, m_{cr} (b)

boundary (Figure 4a) the effect of Ra_V on $\mathrm{Ra}_{B,cr}$ becomes less significant for large values of Ra_H. For example at $\mathrm{Ra}_H = 500$ the critical bioconvection Rayleigh number is almost independent of Ra_V (the corresponding curve in Figure 4a is a straight horizontal line). In contrast, for the rigid top surface (Figure 5a) the effect of Ra_V on $\mathrm{Ra}_{B,cr}$ remains significant even for $\mathrm{Ra}_H = 500$. Figures 4b and 5b indicate that the increase of the horizontal Rayleigh number increases the critical wavenumber for both the stress-free and rigid top boundary cases (except for the large horizontal Rayleigh number case, $\mathrm{Ra}_H = 500$, in which case m_{cr} is almost independent of Ra_V for the stress-free top boundary).

The effect of δ on the critical bioconvection Rayleigh number is displayed in Figure 6a; the results are computed for $\mathrm{Ra}_V = \mathrm{Ra}_H = 100$, $\varpi = 1$, and $\mathrm{Pr} = \mathrm{Sc} = 1$. The increase of δ stabilizes the basic flow. It can be explained as follows. Increased value of D_C corresponds to more uniform oxygen distribution, smaller gradient of oxygen concentration, and, according to Equation (7), less vigorously swimming microorganisms. Since upswimming of microorganisms causes the unstable density stratification, less vigorously swimming microorganisms result in a more stable basic flow. Figure 6b

shows that the increase of δ decreases the critical wavenumber for both types of boundary conditions at the top surface.

The effect of the Schmidt number on the critical bioconvection Rayleigh number is shown in Figure 7a; the results are computed for $\text{Ra}_V = \text{Ra}_H = 100$, $\varpi = 1$, $\delta = 1$, and $\text{Pr} = 1$. The increase of the Schmidt number stabilizes the basic flow; the effect is most pronounced for large values of Sc. The increase of the Schmidt number also generally increases the critical wavenumber (Figure 7b).

The effect of the Prandtl number on the critical bioconvection Rayleigh is shown in Figure 8; the results are computed for $\text{Ra}_V = \text{Ra}_H = 100$, $\varpi = 1$, $\delta = 1$, and $\text{Sc} = 1$. The trend of increasing the Prandtl at lower Pr destabilizes the basic flow, while at larger Prandtl numbers the increase in Pr stabilizes the flow. In Figure 8a the switch from destabilizing to stabilizing occurs for the rigid to surface at around $\text{Pr} = 50$ and for the stress-free case at around $\text{Pr} = 15$. The increase of the Prandtl number also decreases the critical wavenumber.
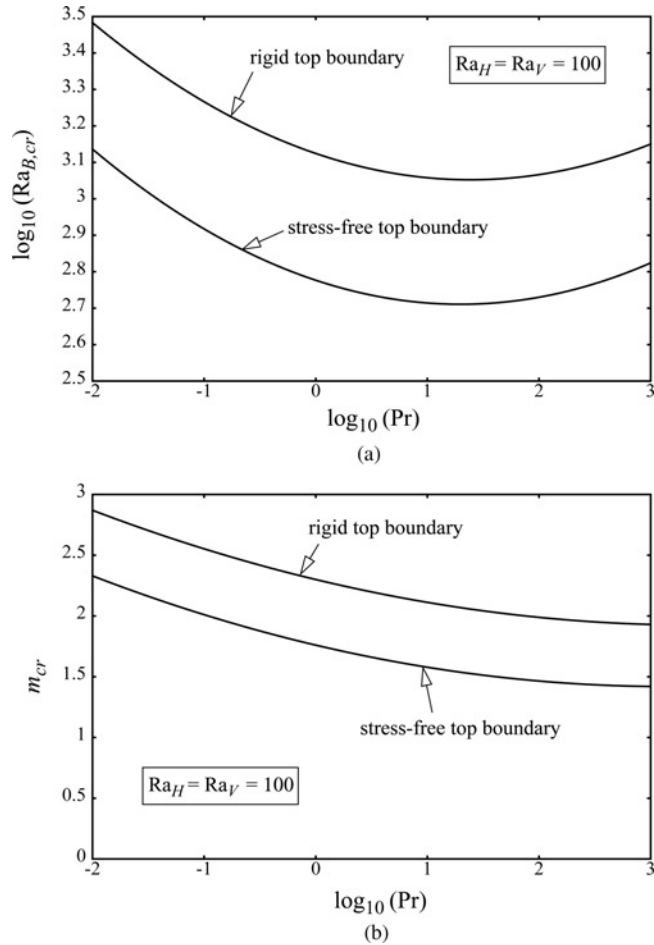
Figure 8.
Effect of Pr on the critical
bioconvection Rayleigh
number, $Ra_{B,cr}$ (a) and on
the critical wavenumber,
m_{cr} (b)

8. Conclusions

Linear stability analysis of the combined bioconvection and thermal instability problem in a horizontal layer of finite depth occupied by a suspension of oxytactic microorganisms in which the basic temperature gradient is inclined to the vertical is carried out.

The increase of the horizontal thermal Rayleigh number (when the vertical thermal Rayleigh number is kept constant) stabilizes the basic flow. The effect of increasing the horizontal thermal Rayleigh number is to distort the basic temperature profile away from the linear one.

The increase of the Schmidt number stabilizes the basic flow. The increase of the Prandtl number first causes the bioconvection Rayleigh number to decrease and then to increase; the dependence $Ra_B(Pr)$ thus exhibits a minimum for both stress-free and rigid top boundaries.

The increase of δ stabilizes the basic flow because an increased value of D_C corresponds to more uniform oxygen distribution, smaller gradient of oxygen concentration, and less vigorously swimming microorganisms.

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